Recall that an undirected graph \( G \) is defined as an ordered pair \((V, E)\) where \( V \) is a finite non-empty set, and \( E \subseteq V^{(2)} \) is a set edges (two-element subsets of \( V \)). A directed graph is defined similarly where the edge set \( E \) is a subset of \( V \times V \setminus \{(v, v) \mid v \in V\} \). The elements of \( V \) are the vertices of \( G \), and the elements of \( E \) are the edges of \( G \). It is convention to refer to \(|V|\) as \( n \) and \(|E|\) as \( m \).

A directed graph \( G \) is defined as an ordered pair \((V, E)\) where \( V \) is a finite non-empty set, and \( E \subseteq V \times V \) is a set of length-two sequences of elements of \( V \). In this class, we further restrict \( E \) not to have elements of the form \((v, v)\) for any \( v \in V \) (no self-loops). The elements of \( V \) are the vertices of \( G \), and the elements of \( E \) are the edges of \( G \). It is convention to refer to \(|V|\) as \( n \) and \(|E|\) as \( m \).

1. Prove that an undirected graph \( G = (V, E) \) is a tree if and only if there exists a unique path between every pair of distinct vertices in \( G \).

**Solution:** First we prove that an undirected graph \( G \) is acyclic if and only if there is at most one path between every pair of distinct vertices in \( G \).

- \((\Leftarrow)\) We prove this implication by proving its contrapositive: if an undirected graph \( G \) contains a cycle (\( G \) is not acyclic), there is some pair of distinct vertices in \( G \) such that there is more than one distinct path between them. To see this, let \( G \) be any undirected graph \( G \) that contains a cycle, and consider two vertices of a cycle in \( G \). Clearly there are two distinct paths between these vertices as given by the edges of that cycle.

- \((\Rightarrow)\) We prove this implication by proving its contrapositive which we prove by smallest counterexample (WOP): for any undirected graph \( G \), if there are at least two distinct paths between a pair of vertices in \( G \), then \( G \) contains a cycle. For sake of contradiction, let \( G \) be an undirected graph with two distinct paths \( P_1 \) and \( P_2 \) between two distinct vertices \( u \) and \( v \) where the number of edges in \( G \) is minimized. If the paths \( P_1 \) and \( P_2 \) only share vertices \( u \) and \( v \), i.e. they are interior-disjoint, then \( P_1 \) concatenated with \( P_2 \) is a cycle, a contradiction.

Henceforth, we assume \( P_1 \) and \( P_2 \) are not interior-disjoint, so they share a vertex \( x \) that is neither \( u \) or \( v \). Let \( P_1^u \) and \( P_1^v \) be the (sub)paths between \( u \) and \( x \) and between \( x \) and \( v \) contained in \( P_1 \), respectively, and let \( P_2^u \) and \( P_2^v \) be the (sub)paths between \( u \) and \( x \) and between \( x \) and \( v \) contained in \( P_2 \), respectively. Note that all of these subpaths must contain at least one edge. There are two cases to consider.

- If \( P_1^u \) and \( P_2^v \) are not equivalent, the subgraph \( G' \) obtained by removing the edges of \( P_2^u \) and \( P_2^v \) from \( G \) has less than \( m \) edges and contains two distinct paths between \( x \) and \( v \), namely \( P_1^v \) and \( P_2^v \), which contradicts the choice of \( G \).

- Otherwise, since \( P_1 \) and \( P_2 \) are not equivalent, it must be that \( P_1^u \) and \( P_2^u \) are not equivalent. Similar to the above, the subgraph \( G' \) obtained by removing the edges of \( P_1^u = P_2^u \) from \( G \) has less than \( m \) edges and contains two distinct paths between \( x \) and \( v \), namely \( P_1^v \) and \( P_2^v \). Again, this contradicts the choice of \( G \).

In either case, a contradiction is reached, so no such \( G \) exists and the claim holds.

With the fact above it is easy to finish the proof. From lecture, we know an undirected graph \( G \) is connected if and only if there is at least one path between every pair of vertices.
Thus, it follows from the above that any undirected graph G is both connected and acyclic if and only if it has both $\geq 1$ path and $\leq 1$ path between every pair of vertices, which is to say there is exactly 1 (unique) path between every pair of vertices. Finally, since we also know an undirected graph is a tree if and only if it is connected and acyclic, we are done by a chain of equivalences.

2. Prove that for any finite partial order $R$ on a finite set $A$, there exists a chain decomposition of $R$ on $A$ of size at most the size of the longest antichain in $R$. (This is one direction of Dilworth’s theorem; we previously argued the converse in lecture.)

**Solution:** Before we begin, we note that the following is a (less concise) proof adapted from one found online here. In recitation, we phrased the proof in terms of reachability in DAGs, specifically Hasse diagrams.

We will give a proof by induction on the size of the set $A$. Let $R$ be a finite strict partial order on a non-empty finite set $A$ (the claim holds vacuously for empty $A$), and let $n$ be the number of pairs in $R$.

Consider the case where $R$ is empty, i.e. $n = 0$. This case is trivial since all pairs of distinct elements are incomparable, and thus the largest antichain is $A$ itself while there is chain decomposition of the same size where each element of $A$ is placed in its own set. Thus, in this case, the claim holds.

Henceforth, we assume $R$ is non-empty (and thus $A$ is non-empty). Assume that the claim holds for all strict partial orders on any set with less than $n$ pairs. Let $s \in A$ be an element where $\neg aRs$ for all $a \in A$ and there exists $b \in A$ such that $sRb$; it can be easily proven such an $s$ always exists. Let $t \in A$ be an element where $sRt$ and $\neg tRa$ for all $a \in A$; again, it can be easily proven such a $t$ must exist. Clearly $C = \{s, t\}$ is a chain since $sRt$.

Now consider the set $A' = A \setminus C$ and relation $R' = R \cap (A' \times A')$; in other words, let $R'$ be the relation obtained by removing all pairs containing $s$ or $t$ from $R$. Note that $R'$ is a strict partial order on set $A'$ by the choices of $s$ and $t$. Let $T'$ be an antichain of $R'$ on $A'$. Since $|R'| < |R|$, there exists a chain decomposition $CD'$ of $R'$ on $A'$ where $|CD'| \leq |T'|$ by the induction hypothesis. This immediately implies a chain decomposition $CD' \cup \{C\}$ of $R$ on $A$ since $C$ is a chain and the only elements of $A$ not in any chains of $CD$ are exactly $s$ and $t$. Now let $T$ be a longest antichain of $R$. There are two cases.

- If $|T'| < |T|$, we have $|CD \cup \{C\}| = |T'| + 1 \leq |T|$, so we are done. Indeed, $CD \cup \{C\}$ is a chain decomposition of $R$ on $A$ of size at most $|T|$ where $T$ is a longest antichain of $R$ on $A$.
- Otherwise, $|T'| \geq |T|$. In this case, the argument of the previous case does not work; we already have at least $|T|$ chains in $CD'$ before adding $C$, so we have to find different chains. To do this, we will (roughly speaking) “split” the set $A$ and relation $R$ into two parts, obtain chain decompositions on each part by the induction hypothesis, then “sew” these together to obtain a single chain decomposition of $R$ on $A$ of the right size, specifically of size $|T|$.
  
  First, we observe that, since any antichain of $R'$ on $A'$ is an antichain of $R$ on $A$, we have $|T'| \leq |T|$. We already have $|T'| \geq |T|$ in this case, so $|T'| = |T|$. Thus, $T'$ is a
longest antichain of $R$ on $A$. Now consider the following sets and relations:

$$A^- = \{ a \in A \mid \exists t \in T'.aRt \}, \quad R^- = R \cap (A^- \times A^-)$$
$$A^+ = \{ a \in A \mid \exists t \in T'.tRa \}, \quad R^+ = R \cap (A^+ \times A^+)$$

In other words, $A^-$ is the set of all elements related to any element of $T'$, and $R^-$ is the subset of pairs of $R$ where both elements are in $A^-$. $A^+$ and $R^+$ are defined similarly.

Before we proceed, let us observe some properties of these sets and relations:

(a) $A^- \cap A^+ = T'$. To see this, suppose for sake of contradiction there is an element $a \in A \setminus T'$ in both $A^+$ and $A^-$. Then there would be elements $x, y \in T'$ such that $xRa$ and $aRy$ by the definitions of $A^+$ and $A^-$, but then $xRy$ by the transitivity of $R$, which contradicts that $T'$ is an antichain of $R$ on $A$.

(b) $A^- \cup A^+ = A$. Indeed, if there was an element $a \in A \setminus T'$ in neither $A^-$ nor $A^+$, then $T' \cup \{a\}$ would be a larger antichain of $R$ on $A$ which is a contradiction.

(c) $|R^-| < |R|$ and $|R^+| < |R|$ since $s \notin A^+$ and $t \notin A^-$.

(d) $T'$ is a longest antichain of both $R^-$ on $A^-$ and $R^+$ on $A^+$; $T'$ is contained in both sets by the first point above, and it is longest since no antichain of these relations can be larger than a longest antichain in $R$ on $A$ such as $T'$ itself.

Together, points (c) and (d) above with the induction hypothesis implies there exists chain decompositions $CD^-$ of $R^-$ on $A^-$ and $CD^+$ of $R^+$ on $A^+$ where $|CD^-|, |CD^+| \leq |T'|$. Now note that every chain in $CD^-$ and $CD^+$ contains exactly one element of $T'$, otherwise there would be a chain with two elements of $T'$ which contradicts that $T'$ is an antichain. Furthermore, this implies $|CD^-|, |CD^+| \geq |T'|$, so now we have $|CD^-| = |CD^+| = |T'|$ where every chain in these decompositions contains exactly one element of $T'$.

Finally, for each $x \in T'$, define $C^-_x$ to be the chain in $CD^-$ that contains $x$, and define $C^+_x$ similarly. These are the chains that we “sew” together. Note that $C^-_x \cup C^+_x$ is a chain of $R$ on $A$ by the transitivity of $R$ and definitions of $A^-$ and $A^+$, so $\{C^-_x \cup C^+_x \mid x \in T'\}$ is a chain decomposition of $R$ on $A$ of size $|T|$ as desired.

3. A tournament graph is a directed graph where exactly one of $(u, v)$ or $(v, u)$ is in $E$ for every pair of distinct vertices $u, v \in V$. A champion of the graph is a vertex from which every other vertex is reachable by a path of length at most two from the champion. That is, every other vertex is an out-neighbor of the champion, or it is the out-neighbor of an out-neighbor of the champion. Prove that any vertex in $G$ with largest out-degree is a champion.

Solution: Suppose for sake of contradiction a vertex $u$ with maximum out-degree is not a champion, and let $X$ be the set of out-neighbors of $u$. Since $u$ is not a champion, there is some vertex $v$ that $u$ cannot reach via paths of length at most two; in particular, $v$ is not in $X$, and $v$ is not an out-neighbor of any vertex in $X$. In other words, $(x, v) \notin E$ for every $x \in X \cup \{u\}$. Since $G$ is a tournament graph, we have $(v, x) \in E$ for every $x \in X \cup \{u\}$. It follows that the out-degree of $v$ is at least one more than $|X|$, the out-degree of $u$, which is a contradiction.