1 Induction

1. Induction Hypothesis Let \( P(n) : \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \) be true for all natural numbers \( k \leq n \in N \).

We wish to prove \( P(n + 1) \) holds true,

Base Case \( P(1) : \sum_{i=0}^{1} 2^i = 1 + 2 = 3 = 4 - 1 = 2^{1+1} - 1 \) holds true.

Induction Step

\[
\sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1 \quad (\text{from the Induction Hypothesis})
\]

\[
= 2^{k+2} - 1
\]

Thus, we show that \( P(n+1) \) holds whenever \( P(n) \) holds. Thus, by the principle of Mathematical Induction, \( P(n) \) holds for all natural numbers \( n \).

2. Induction Hypothesis Let \( P(n) : \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} \) be true for all natural numbers \( k \leq n \in N \).

We wish to prove \( P(n + 1) \) holds true,

Base Case \( P(1) : \sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2+1)}{6} \) holds true.

Induction Step

\[
\sum_{i=0}^{k+1} i^2 = (k + 1)^2 + \sum_{i=0}^{k} i^2
\]

\[
= (k + 1)^2 + \frac{k(k+1)(k+2)}{6} \quad (\text{from the Induction Hypothesis})
\]

\[
= \frac{(k+1)(k+1 + k(k+2))}{6}
\]

\[
= \frac{(k+1)(k+2)(2k+3)}{6}
\]

\[
= \frac{(k + 1)((k + 1) + 1)(2(k + 1) + 1)}{6}
\]
Thus, we show that \( P(n+1) \) holds whenever \( P(n) \) holds. Thus, by the principle of Mathematical Induction, \( P(n) \) holds for all natural numbers \( n \).

2 Euclid’s Algorithm

(a) For any integer \( x \), \( x \) divides \( 0 \), because \( x \times 0 = 0 \), therefore, \( a \) is a factor of \( 0 \) (written as \( a | 0 \)) and since, the greatest factor of \( a \) is \( a \), therefore, \( \text{GCD}(a, 0) = a \).

(b) If, \( a \leq b \) \( \Rightarrow \) \( \text{GCD}(b, a) = \text{GCD}(b, a \mod b) \),

Else if \( a > b \), then let \( a\mod b = a - kb \), where \( k \) is the quotient when \( a \) is divided by \( b \).

Let \( c \) be a common divisor of \( a \) and \( b \). \( \Rightarrow c|a, c|b \Rightarrow (a\mod b)/c = a/c - k \times (b/c) \) is an integer because each term is an integer. \( \Rightarrow c|(a\mod b) \)

Also, if \( c \) is a common divisor for \( b \) and \( a \mod b \) \( \Rightarrow c|b, c|(a\mod b) \Rightarrow a/c = k \times b/c + (a\mod b)/c \) is an integer, because all terms in the expansion are integers.

Thus, all common divisors of \((a, b)\) and \((b, a\mod b)\) are identical \( \Rightarrow \text{GCD}(a, b) = \text{GCD}(a\mod b, b) \)

(c) Case I: If \( a < 2b \), then \((b + a\mod b) \leq (a + b) - b \leq \frac{2}{3}(a + b) \)

Case II: If \( a \geq 2b \), then \((b + a\mod b) \leq 2b \leq \frac{2}{3}(a + b) \)

Thus, the value of \((a + b)\) reduces by a factor of at least \( \frac{2}{3} \) in each step.

T(a+b): Running time of the algorithm when the input is \((a, b)\)

Induction Hypothesis \( P(N) \): \( T(a + b) \leq \log_{\frac{2}{3}}(a + b) + k \), for some large \( k \), to satisfy the base case.

is true for all \( a + b \leq N \)

Induction Step

\[
T(a + b + 1) = 1 + T\left(\frac{2}{3}(a + b + 1)\right)
\]

\[
\leq 1 + \log_{\frac{2}{3}}\left(\frac{2}{3}(a + b + 1)\right) + k \quad \text{(From the Induction Hypothesis)}
\]

\[
= 1 + \log_{\frac{2}{3}}(a + b + 1) - 1 + k \quad \text{\((log(ab) = log(a) + log(b))\)}
\]

\[
= \log_{\frac{2}{3}}(a + b + 1) + k
\]

Hence, by the principle of Mathematical Induction, \( P(a+b) \) holds true for all naturals \( a, b \). Thus, \( T(a + b) = O(\log(a + b)) \).