COMPSCI330 Design and Analysis of Algorithms
Assignment 7

Due Date: Thursday March 28, 2019

Guidelines

- **Describing Algorithms** If you are asked to provide an algorithm, you should clearly define each step of the procedure, establish its correctness, and then analyze its overall running time. There is no need to write pseudo-code; an unambiguous description of your algorithm in plain text will suffice. If the running time of your algorithm is worse than the suggested running time, you might receive partial credits.
- **Typesetting and Submission** Please submit the problems to GradeScope. You will be asked to **label your solution for individual problems**. Failing to label your solution can cost you 5% of the total points (3 points out of 60 for this homework).
- LaTeX is preferred, but answers typed with other software and converted to pdf is also accepted. Please make sure you submit to the correct problem, and your file can be opened by standard pdf reader. **Handwritten answers or pdf files that cannot be opened will not be graded.**
- **Timing** Please start early. The problems are difficult and they can take hours to solve. The time you spend on finding the proof can be much longer than the time to write. If you need more time for your homework please use this form and submit a STINF.
- **Collaboration Policy** Please check this page for the collaboration policy. You are **not** allowed to discuss homework problems in groups of more than 3 students. **Failure to adhere to these guidelines will be promptly reported to the relevant authority without exception.**

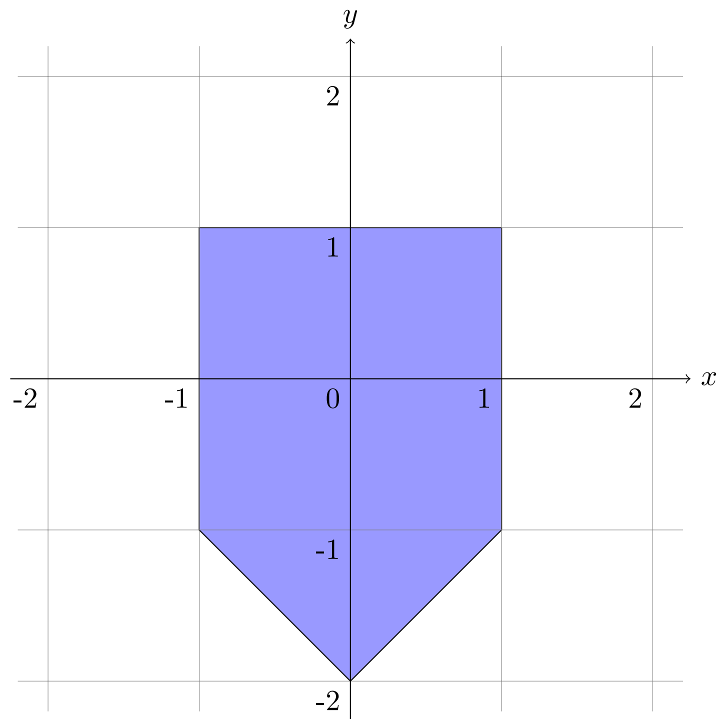

Figure 1: Feasible Region of a Linear Program

Problem 1 (Linear Programming Basics)**.** (15 points)

Consider a linear program with two variables x and y. The set of feasible solutions is the blue region in Figure 1.

(a) (10 points) Write out the constraints for x, y so that the feasible region is exactly equal to the blue region in Figure 1.

(b) (5 points) If the objective function is $\max x + 2y$, what is the optimal solution? If the objective function is $\max x + y$, what is the optimal solution?

Problem 2 (Fractional Knapsack revisited)**.** (20 points) Linear programs are not only useful for graph problems. In this problem we will use linear programming to solve variants of fractional knapsack.

Recall the setting of the fractional knapsack problem: there are n items. Item i has weight $w_i \geq 0$ and value $v_i \geq 0$. We have a knapsack with capacity W. The goal is to put items into the knapsack so that the total weight is no more than the capacity and the total value is maximized. It is possible to put a fraction p of an item i, which will contribute weight $p \cdot w_i$ and value $p \cdot v_i$.

(a) (10 points) Suppose each item has has a volume s_i, and the knapsack has total volume S. Write a linear program that computes the optimal way of putting items into the knapsack, such that the total weight is no more than capacity W, total volume is less than the volume S, and total value is maximized. Again, a fraction p of item i will contribute $p \cdot s_i$ to total volume.

(b) (10 points) Now suppose the knapsack is shared between k different people who have different opinions about the values of items. For person j ($j = 1, 2, ..., k$), the value of item i is $v_i^j \geq 0$. For each way of putting items into the knapsack, different people will have different evaluations for the knapsack (p fraction of item i will contribute $p \cdot v_i^j$ to the evaluation of person j). We want to be fair to all the people, so we will define the actual value of the knapsack to be the minimum evaluation out of the k people. Write a linear program that finds the maximum possible value of a knapsack. (Note: this problem is separate from (a), so there are no volume constraints.)

Problem 3 (Duality of Shortest Path)**.** (25 points) In class we used a linear program with variables on vertices to solve the shortest path problem. Given a directed graph $G = (V, E)$, a starting vertex s and an ending vertex t, see below for an equivalent version of the LP that we did in class (in this problem you can assume all the edge weights $w(u, v) \geq 0$, and there exists a path from s to t)

$$\begin{aligned} \max \quad & x_t - x_s \\ \forall (u, v) \in E \quad x_v &\leq x_u + w(u, v). \\ \forall u \in V \quad x_u &\geq 0. \end{aligned}$$

(a) (15 points) Write the dual of this linear program. (Hint: This is a maximization problem with "$\leq$" constraints, so it looks more similar to the canonical form of the dual LP that we covered in class. The dual of dual LP is the original primal LP - which would be a minimization problem.)
(b) (5 points) Suppose $(s, u_1, u_2, ..., u_k, t)$ is a shortest path from s to t, give a feasible solution for the dual LP you wrote in (a), whose value is equal to the length of the path.
(c) (5 points) Prove that the optimal values for both the primal and dual LP are equal to the shortest path distance from s to t.