

Lecture 16: Linear Programming I

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1 Overview

In this lecture, we will learn the basic knowledge of **Linear Programming (LP)**, which is optimizing a linear function (objective function) under a set of linear inequality constraints. Specifically, we will first define linear programming, then discuss the intuition about what it means to "solve" an LP. After that, we will learn how to think of the constraints and how to solve an LP. Finally, we talk about applying LP to other problems, i.e., formalizing other problems into an LP instance.

2 Basics for LP

2.1 Definition of LP

Definition 1. A linear program has n variables x_1, x_2, \dots, x_n (represented as a vector $x \in \mathbb{R}^n$), and objective function $f(x) = c^\top x = \sum_{i=1}^n c_i x_i$ (the c_i 's are known), and a set of linear constraints.

Also, the goal of the linear program is to maximize/minimize the objective function given the constraints.

2.2 Example LP

Here we provide an example of LP:

$$\begin{array}{ll} \max & 2x_1 + x_2 \\ \text{subject to} & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 + x_2 \leq 1 \end{array}$$

In this example, the variables are x_1 and x_2 , the objective function is $f(x) = 2x_1 + x_2$ ($c = (2, 1)$), and the constraints are the three inequalities, i.e., $x_1 \geq 0$, $x_2 \geq 0$, and $x_1 + x_2 \leq 1$.

2.3 Solution to an LP

2.3.1 Feasible solution

Definition 2. A **feasible solution** is a vector x that satisfies all the constraints.

For example, for the example LP above, $x = (0, 0)$ is a feasible solution because $x_1 = 0$ and $x_2 = 0$ satisfies all the constraints, and the corresponding objective function value $f(x) = 2 \times 0 + 0 = 0$. Similarly, $x = (0, 1)$ is a feasible solution with objective value $f(x) = 1$.

However, $x = (1, 1)$ is not a feasible solution because $x_1 + x_2 = 2 > 1$, which contradicts the last constraint.

2.3.2 Optimal solution

Definition 3. An **optimal solution** is a vector x that optimizes the objective while satisfying all the constraints.

For example, for the example LP above, the optimal solution is $x = (1, 0)$ with $f(x) = 2x_1 + x_2 = 2$.

Also, the **value of the optimal solution** is the value $f(x)$ for the optimal solution x . Sometimes the value of the optimal solution is also referred to as “optimal solution”.

3 Geometric Interpretation

To get the intuition about how we think of the constraints and how we solve an LP, we will interpret LP from a geometric perspective. The geometric interpretation for the example LP above is shown in Figure 1.

3.1 Constraints

The constraints of an LP correspond to **half-planes** (half-spaces in higher dimensions). For example, the constraint $x_2 \geq 0$ corresponds to the half-plane above x_1 -axis, i.e., the green part in Figure 1. Also, the constraint $x_1 + x_2 \leq 1$ corresponds to the half-plane that is below the line $x_1 + x_2 = 1$, i.e., the red part in 1.

There are two steps for determining the corresponding half-plane for a certain constraint: First, we turn the inequality into equality and draw the corresponding line, e.g., for constraint $x_1 + x_2 \leq 1$, we draw the line $x_1 + x_2 = 1$. After that, we select a point outside that line to determine which side is the correct half-space, e.g., we select the origin and compute $x_1 + x_2 = 0 + 0 = 0 \leq 1$, so the half-plane below $x_1 + x_2 = 1$ is the half-plane we want.

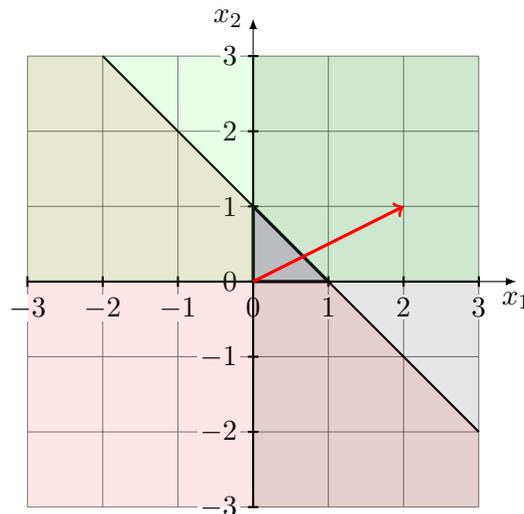


Figure 1: Geometric Interpretation for Example LP

3.2 System of Linear Inequalities

In the original LP, there are a system of linear inequalities, and we want all of them to hold. Thus, after finding all the half-planes corresponding to them, we take the **intersection of those half-planes** and get a polygon (polytope in higher dimensions). In the example in Figure 1, the polygon we find is a triangle, which is colored blue in the figure. This polygon we get is also called **feasible region** because it contains all the feasible solutions of this LP.

3.3 Objective function

The objective function can be interpreted as the **direction of gravity**. The direction of the vector connecting the origin with the vector c is defined as the direction of gravity. In the example LP, $c = (2, 1)$, so the direction of gravity is the red arrow in Figure 1. If we rotate the feasible region such that the direction of gravity is pointing downwards, the optimal solution is the lowest point in the feasible region. In this particular example, the lowest point is $(1, 0)$, so it is the optimal solution of the example LP.

4 Canonical Form

An LP can always be written in many different ways. For simplicity of analysis and interacting with LP solvers, we need a standard way of writing LP, which is called the canonical form. The canonical form is defined as:

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

This form is written in a linear algebraic way, to transfer it to the usual way that an LP is written, we note that

$$\begin{aligned} \langle c, x \rangle &= c^\top x = \sum_{i=1}^n c_i x_i \\ Ax \geq b &\Leftrightarrow \forall j \in \{1, 2, \dots, m\}, \sum_{i=1}^n A_{j,i} x_i \geq b_j \\ x \geq 0 &\Leftrightarrow \forall i \in \{1, 2, \dots, n\}, x_i \geq 0. \end{aligned}$$

Take the example LP as an example, we transfer it to the canonical form:

$$\begin{aligned} \max \quad & 2x_1 + x_2 & \min \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 \geq 0 & \text{subject to} \quad & -x_1 - x_2 \geq -1 \\ & x_2 \geq 0 & & x_1 \geq 0 \\ & x_1 + x_2 \leq 1 & & x_2 \geq 0 \end{aligned} \quad \Longrightarrow$$

In this case, we have $A = (-1, -1)$, $b = -1$, and $c = (-2, -1)$.

4.1 Methods to transform an arbitrary LP to canonical form

4.1.1 Splitting Equalities

Equalities are not allowed in canonical form. Every time we have an equality, e.g., $x = 1$, we can split it into two inequalities: $x \geq 1$ and $x \leq 1$.

4.1.2 Flipping Inequality Sign / Objective Function

When we have incorrect inequality sign, e.g., we have a constraint $x_1 + x_2 \leq 1$, but we want a \geq sign, we multiply -1 to both sides of the inequality to flip the sign. Thus, we get $-x_1 - x_2 \geq -1$.

The same thing happens for the objective function: When we have a maximize objective but want a minimize instead, we can multiply -1 to the objective function. For example, $\max 2x_1 + x_2$ is equivalent to $\min -2x_1 - x_2$.

4.1.3 Dealing with Free Variables

Definition 4. Call x a **free variable** if there is no constraint $x \geq 0$.

Since the canonical form doesn't allow free variables, we do the change of variables $x = x_1 - x_2$, where x is the original free variable and x_1 and x_2 are the variables we add, such that $x_1 \geq 0$ and $x_2 \geq 0$. Take the following LP as an example:

$$\begin{array}{ll} \max & x + y \\ \text{subject to} & y \geq 0 \\ & x + 2y \leq 3 \\ & x \geq -5 \end{array}$$

Let $x = x_1 - x_2$ (also apply the two techniques mentioned above) and we get

$$\begin{array}{ll} \min & -x - y \\ \text{subject to} & -x_1 + x_2 - 2y \geq -3 \\ & x_1 - x_2 \geq -5 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & y \geq 0 \end{array}$$

or equivalently,

$$\begin{array}{ll} \min & \langle (-1, 1, -1), (x_1, x_2, y) \rangle \\ \text{subject to} & \begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \geq \begin{pmatrix} -3 \\ -5 \end{pmatrix} \\ & \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \geq 0 \end{array}$$

5 Using LP to solve graph problems

5.1 Example: Bipartite Matching

In this example, we are going to form an LP instance for bipartite maximum matching problem, i.e., there are n courses and n classrooms, and each course can only be held in certain classrooms. A classroom can only hold one course, and a course only need one classroom. Given those constraints, we want to find the maximum possible number of courses that can be held at the same time.

Then we follow the steps to form an LP for this problem.

5.1.1 Find the Variables

In graph problems, the variables are usually related to the edges or vertices. In this particular problem, we want to know for each edge if it's in the matching or not, so we define (i is the course, j is the classroom)

$$x_{i,j} = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge in matching} \\ 0 & \text{if } (i,j) \text{ is not in matching} \end{cases} .$$

5.1.2 Write Constraints

There are three kinds of constraints:

$$\begin{array}{ll} \text{Every classroom can hold at most 1 course:} & \forall j \in \{1, 2, \dots, n\}, \sum_{i=1}^n x_{i,j} \leq 1 \\ \text{Every course needs only at most 1 classroom:} & \forall i \in \{1, 2, \dots, n\}, \sum_{j=1}^n x_{i,j} \leq 1 \\ x_{i,j} \text{ should be 0 or 1:} & \forall i, j \in \{1, 2, \dots, n\}, 0 \leq x_{i,j} \leq 1 \end{array}$$

Note that in an LP we cannot constrain $x_{i,j}$ to be 0 or 1 because this is not linear. Thus, we try the best approximation, but there is a risk that $x_{i,j}$ may not be an integer, e.g., $x_{i,j} = 0.5$.

5.1.3 Write the Objective

In this problem, we want to maximize the number of matched edges, that is,

$$\max \sum_{(i,j) \in E} x_{i,j}$$

5.2 Options for writing Bipartite matching as an LP

There are two options for writing Bipartite matching as an LP:

- Have $x_{i,j}$ variables for all (i,j) pairs, so the only place where graph influences the LP is in the objective
- Only have $x_{i,j}$ variables if (i,j) is an edge in the graph

The first option is easy to write, but there are many redundant variables because the non-edge variables are not in the objective and setting all those non-edge variables to 0 always gives us better solutions than not setting them to 0. This makes the LP much slower to solve.

The second option is a bit hard to write because the constraints need to be changed to $\forall i, \sum_{j:(i,j) \in E} x_{i,j} \leq 1$, $\forall j, \sum_{i:(i,j) \in E} x_{i,j} \leq 1$, and $\forall (i,j) \in E, 0 \leq x_{i,j} \leq 1$. However, this LP is usually solved quicker than the first one in practice.

6 Fractional Solution versus Integral Solution

Definition 5. If **all** variables are integers, the solution is called an **integral solution**. Otherwise, the solution is called a **fractional solution**.

For LP, the variables can be fractions in general, which may cause us trouble because we usually want integral solutions. However, for this matching LP, it's actually possible to prove there exists integral optimal solution.