Section: Properties of Regular Languages

Example

\[ L = \{ a^n b a^n \mid n > 0 \} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
\[ L = \{ x \mid x \text{ is a positive even integer} \} \]

\[ L \text{ is closed under} \]

- addition? \(\text{yes}\)
- multiplication? \(\text{yes}\)
- subtraction? \(\text{no}\)
- division? \(\text{no}\) \(2 - 4 = -2\)

**Closure of Regular Languages**

**Theorem 4.1** If \(L_1\) and \(L_2\) are regular languages, then

\[
L_1 \cup L_2 \\
L_1 \cap L_2 \\
L_1 L_2 \\
\bar{L}_1 \\
L_1^* \\
\]

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages

$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.

$L_1 = L(r_1)$ and $L_2 = L(r_2)$

$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$

$\Rightarrow$ closed under union

$r_1r_2$ is r.e. denoting $L_1L_2$

$\Rightarrow$ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$

$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

- Final states in $M$
  - are nonfinal states
    - in $M'$

- Nonfinal states in $M$
  - are final states in $M$

Show $\not\exists L(M') \Rightarrow \exists \omega \\
\Rightarrow \exists$ closed under complementation.
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1= (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2= (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M'=(Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = Q \times P$

$\delta'$:

$\delta'(q_i, p_i, a) = (q_k, p_e)$ if

$\delta_1((q_i, a) = q_k) \in M_1$ and

$\delta_2((p_i, a) = p_e) \in M_2$.

$F' = \{ (q_i, p_i) \in Q' | q_i \in F_1 \land p_i \in F_2 \}$
Example:
Regular languages are closed under

reversal \quad L^R

difference \quad L_1 - L_2

right quotient \quad L_1 / L_2

homomorphism \quad h(L)
Right quotient

Def: \( L_1/L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\( L_1 = \{ a^* b^* \cup b^* a^* \} \)
\( L_2 = \{ b^n | n \text{ is even, } n > 0 \} \)
\( L_1/L_2 = \)
try every state as a start state. Can you get a string from $L_2$?
Theorem If \( L_1 \) and \( L_2 \) are regular, then \( L_1/L_2 \) is regular.

Proof (sketch)

\[ \exists \text{ DFA } M = (Q, \Sigma, \delta, q_0, F) \text{ s.t. } L_1 = L(M). \]

Construct DFA \( M' = (Q, \Sigma, \delta, q_0, F') \)

For each state \( i \) do

Make \( i \) the start state (representing \( L_i' \))

\[ \text{if } L_1 \cap L_2 \neq \emptyset \]

\[ \text{put } q_i \text{ in } F' \text{ in } M' \]

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$h(bc) = 000$

$h(ab^*) = 11(00)^*$
Questions about regular languages:
L is a regular language.

• Given L, Σ, w ∈ Σ*, is w ∈ L?
  
  Construct DFA test to see if it accepts w

• Is L empty?

  DFS

• Is L infinite?

  Check for cycle on path from start state to final state

• Does L₁ = L₂?

  (L₁ ∪ L₂) ∩ (L₁ ∩ L₂) = φ
  equivalent
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular? **Yes**

If $L$ is infinite, is $L$ regular? **Maybe**

- $L_1 = \{a^n b^m | n > 0, m > 0\}$
- $L_2 = \{a^n b^n | n > 0\}$ **Not**
Prove that $L_2 = \{a^n b^n | n > 0\}$ is not regular.

- Proof: Suppose $L_2$ is regular.
  $\Rightarrow \exists$ DFA $M$ that recognizes $L_2$

Consider a long string $a^k b^k \in L_2$

3 states

$M$ has a finite number of states, $K$ states

with $K$ states

a loop in the $a$'s
Some loop in the $a^3$

say $a^3$ in the loop

$\Rightarrow$

$a^3b^k$ is accepted

$\Rightarrow a^{k+1}b^k$ is also accepted

$a^{k+1}b^k \notin L_2$!

Contradiction. DFA doesn't exist.
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$|xy| \leq m$$
$$|y| \geq 1$$
$$xy^iz \in L \text{ for all } i \geq 0$$
To Use the Pumping Lemma to prove L is not regular:

- Proof by Contradiction.
  Assume L is regular.
  ⇒ L satisfies the pumping lemma.
  Choose a long string w in L, |w| ≥ m.

Show that there is NO division of w into xyz (must consider all possible divisions) such that |xy| ≤ m, |y| ≥ 1 and xy^i z ∈ L ∀ i ≥ 0.

The pumping lemma does not hold. Contradiction!
⇒ L is not regular. QED.
Example \( L = \{a^n c b^n | n > 0 \} \)

\( L \) is not regular.

- **Proof:**

Assume \( L \) is regular.

⇒ the pumping lemma holds.

Choose \( w = a^m c b^m \)
It should be

\[ xy^i z \in L \Rightarrow a \geq 0 \]

\[ y^i = 2x = 2y \]

Contradiction.

\[ \Rightarrow L \text{ is not regular} \]
Example $L = \{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

• Proof:

Assume $L$ is regular.

⇒ the pumping lemma holds.

Choose $w =$

So the partition is:

$$w = xyz$$

$$x = a^k$$

$$y = a^{j}$$

$$z = a^{m-k-j} b^{m+s} c^s$$

$$i = 0$$

$$m-i \leq k$$

$$\text{num a's} \neq \text{num a's} + \text{num c's}$$
Example \( \Sigma = \{a, b\} \), 
\( L = \{ w \in \Sigma^* \mid n_a(w) > n_b(w) \} \)

\( L \) is not regular.

• Proof:
  
  Assume \( L \) is regular.
  
  \( \Rightarrow \) the pumping lemma holds.
  
  Choose \( w = b^m a^m + 1 \)
  
  So the partition is:

\[
\begin{align*}
  x &= b^k, \quad y = b^j, \quad z \geq 0 \\
  m - k - j &\geq m + 1 \\
  z &= b^{m - k - j} a^j \\

  \forall i \in \mathbb{N}, \quad xy^i z \in L \\
  i = 2 \\
  xy^2 z = b^m a^m + 1 \\
  \text{num of } a's \leq \text{num of } b's
\end{align*}
\]
Example $L = \{ a^3 b^n c^{n-3} | n > 3 \}$
(shown in detail on handout)
$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- **Proof Outline:**
  Assume $L$ is regular.
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  closure properties $\Rightarrow L'$ is regular. Contradiction!
  $L$ is not regular. QED.
Example \( L = \{ a^3 b^n c^{n-3} | n > 3 \} \)

L is not regular.

- Proof: (proof by contradiction)
  Assume \( L \) is regular.
  Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
  \( h(a) = a \quad h(b) = a \quad h(c) = b \)
  \( h(L) = \{ a^{n+3} b^{n-3} | n > 3 \} \)

is regular

\[ L' = h(L) = \{ a^{n+3} b^{n+3} | n > 3 \} \]

\[ L'' = \{ ab, aabb, a^3 b^3, 414 \} \]

\[ L' \cup L'' = \{ a^m b^n | n > 3 \} \]

is regular

Contradiction, \( L \) is not regular.
Example \( L = \{ a^n b^m a^m | m \geq 0, n \geq 0 \} \)

\( L \) is not regular.

- **Proof**: (proof by contradiction)
  
  Assume \( L \) is regular.
Example: $L_1 = \{a^n b^n a^n | n > 0\}$

$L_1$ is not regular.