Section: Properties of Regular Languages

Example

\[ L = \{a^nba^n \mid n > 0\} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
\( L = \{ x \mid x \text{ is a positive even integer} \} \)

\( L \) is closed under

- addition? \( \text{yes} \)
- multiplication? \( \text{yes} \)
- subtraction? \( \text{no} \)
- division? \( \text{no} \)

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

\[
\begin{align*}
L_1 \cup L_2 \\
L_1 \cap L_2 \\
L_1 L_2 \\
\overline{L_1} \\
L_1^*
\end{align*}
\]

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
⇒ $\exists$ reg. expr. $r_1$ and $r_2$ s.t.
  $L_1 = L(r_1)$ and $L_2 = L(r_2)$
  $r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
  ⇒ closed under union
  $r_1 r_2$ is r.e. denoting $L_1 L_2$
  ⇒ closed under concatenation
  $r_1^*$ is r.e. denoting $L_1^*$
  ⇒ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

- Final states in $M$ are nonfinal states in $M'$
- Nonfinal states in $M$ are final states in $M'$

Show $w \notin L(M_1) \Rightarrow w \in L$ is closed under complementation.
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = \varnothing \times P$

$\delta'$:

$\delta'( ((q_i, p_i), a) ) = (q_k, p_e)$ if

$\delta_1((q_i, a) = q_k) \in M_1$ and

$\delta_2((p_i, a) = p_e) \in M_2$

$F' = \{ (q_i, p_i) \in Q' \mid q_i \in F_1 \land p_i \in F_2 \}$
Example:
Regular languages are closed under

- reversal \( L^R \)
- difference \( L_1 - L_2 \)
- right quotient \( L_1 / L_2 \)
- homomorphism \( h(L) \)
Right quotient

Def: \( L_1/L_2 = \{ x \mid xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[ L_1 = \{ a^*b^* \cup b^*a^* \} \]
\[ L_2 = \{ b^n \mid n \text{ is even, } n > 0 \} \]
\[ L_1/L_2 = \]
idea

Try every state as a start state. Can you get a string from L_2?
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=(Q,\Sigma,\delta,q_0,F)$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q,\Sigma,\delta,q_0,F')$

For each state $i$ do

Make $i$ the start state (representing $L'_i$)

If $L'_i \cap L_2 \neq \emptyset$

put $q_i$ in $F'$ in $M'$

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$h(bc) = 000$

$h(ab^*) = 10000^*$
Questions about regular languages:
L is a regular language.

- Given L, Σ, w ∈ Σ*, is w ∈ L?

  Construct DFA test to see if it accepts w

- Is L empty?

  DFS

- Is L infinite?

  Check for cycle on path from start state to final state

- Does L₁ = L₂?

  \((L₁ \cup L₂) \cap (L₁ \cap L₂) = \emptyset\)

  equivalent
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular? Yes

If $L$ is infinite, is $L$ regular? Maybe

- $L_1 = \{a^n b^m | n > 0, m > 0\}$ = aabb
- $L_2 = \{a^n b^n | n > 0\}$ Not
Prove that \( L_2 = \{a^n b^n | n > 0 \} \) is not regular.

- Proof: Suppose \( L_2 \) is regular.
  \( \Rightarrow \exists \) DFA M that recognizes \( L_2 \)

M has a finite number of states, \( K \) states

Consider a long string \( a^K b^K \in L_2 \)

With \( K \) states, there must be a loop in the a's
Some loop in the d's, say \( a^3 \) in the loop

\[ \rightarrow 0 \]

\[ \rightarrow 0 \]

\[ \rightarrow 0 \]

\[ \rightarrow 0 \]

\[ \rightarrow 0 \]

\[ a < b^k \] is accepted

\[ \Rightarrow a^{k+1} b^k \] is also accepted

\[ a^{k+1} b^k \notin \Sigma^2 \]

Contradiction. DFA doesn't exist.
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$|xy| \leq m$$
$$|y| \geq 1$$
$$xy^iz \in L \text{ for all } i \geq 0$$
To Use the Pumping Lemma to prove L is not regular:

- Proof by Contradiction.
  Assume L is regular.
  ⇒ L satisfies the pumping lemma.
  Choose a long string \( w \) in L, \( |w| \geq m \).
  Show that there is NO division of \( w \) into \( xyz \) (must consider all possible divisions) such that \( |xy| \leq m, |y| \geq 1 \) and \( xy^iz \in L \ \forall \ i \geq 0 \).
  The pumping lemma does not hold. Contradiction!
  ⇒ L is not regular. QED.
Example $L = \{a^n cb^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  ⇒ the pumping lemma holds.
  
  Choose $w = a^m cb^m$
  
  only way to partition it into three parts $xyz$
  
  $x = a^k$, $y = a^j$, $j > 0$
  
  $m - k - j c b^m$
  
  $z = a^{m-k-j} cb^m$
it should be

$$xy^iz \leq 1$$

$$a \geq 0$$

$$x^y = 2x = 2y$$

Contradiction.

$$\Rightarrow L$$ is not regular.
Example $L=\{a^nb^{n+s}c^s|n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w=a^{rn}b^{m+s}c^s$
  
  So the partition is:
  
  $w=xyz$ \[P.L.\]
  
  $|xy| \leq m$ \[\Rightarrow\]
  
  $i=0, m-1 \leq m+s \leq k$ \[\Rightarrow\]
  
  $\text{num } a^s \neq \text{num } a^s + \text{num } c^s$
Example $\Sigma = \{a, b\}$,
$L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

- **Proof:**

  Assume $L$ is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w = b^m a^m + 1$

  So the partition is:

  $x = b^i$, $y = b^j$, $i > 0$

  $z = b^{m-k-j} a^{m+1}$

  $\forall i \, x y^i z \in L$

  $i = 2 \, x y z = b^9 a^4 \notin L$

  num of $a's \leq \text{num of } b's$
Example $L = \{a^3b^nc^{n-3} | n > 3\}$
(shown in detail on handout)
$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- **Proof Outline:**
  
  Assume $L$ is regular.
  
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  
  closure properties $\Rightarrow L'$ is regular. Contradiction!
  
  $L$ is not regular. QED.
Example $L = \{a^3b^n c^{n-3} | n > 3\}$

$L$ is not regular.

- Proof: (proof by contradiction)
  Assume $L$ is regular.
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  $h(a) = a \  h(b) = a \  h(c) = b$
  $h(L) = \{a^n b^{n-3} | n > 3\}$ is regular.
Example $L = \{a^n b^m a^m | m \geq 0, n \geq 0\}$

$L$ is not regular.

- Proof: (proof by contradiction)
  Assume $L$ is regular.

$$L_1 = b b^* a^*$$

$$L_2 = L \cap \{b^n a | n \geq 0\}$$

Define homomorphism $h: \Sigma \rightarrow \Sigma^*$

$$h(a) = b \quad h(b) = a$$

$$h(L_2) = \{a^n b^n | n \geq 0\}$$

Showed that $L$ is not regular

$\Rightarrow L$ is not regular.
Example: \( L_1 = \{ a^n b^n a^n \mid n > 0 \} \)

\( L_1 \) is not regular.

Proof

Assume \( L_1 \) is regular

Let \( L_2 = \{ a^n \} \) \( L_2 \) is regular right quotient

\( L_3 = L_1 \setminus L_2 = \{ a^n a^n \mid n > 0, 0 \leq p \leq n \} \)

\( L_4 = L_3 \cap \{ a b b \} = \{ a b^n \mid n > 0 \} \)

should be regular

But shown not regular

Contradiction!

\( \Rightarrow L_1 \) is not regular