Definition: A language L is recursively enumerable if there exists a TM M such that L = L(M).

Definition: A language L is recursive if there exists a TM M such that L = L(M) and M halts on every w ∈ Σ⁺.

Enumeration procedure for recursive languages

To enumerate all w ∈ Σ⁺ in a recursive language L:

- Let M be a TM that recognizes L, L = L(M).
- Construct 2-tape TM M’
  - Tape 1 will enumerate the strings in Σ⁺
  - Tape 2 will enumerate the strings in L.
  - On tape 1 generate the next string v in Σ⁺
  - simulate M on v
    - if M accepts v, then write v on tape 2.
Enumeration procedure for recursively enumerable languages

To enumerate all \( w \in \Sigma^+ \) in a recursively enumerable language \( L \):

Repeat forever

- Generate next string (Suppose \( k \) strings have been generated: \( w_1, w_2, ..., w_k \))
- Run M for one step on \( w_k \)
  - Run M for two steps on \( w_{k-1} \)
  - ...
  - Run M for \( k \) steps on \( w_1 \).
- If any of the strings are accepted then write them to tape 2.

**Theorem** Let \( S \) be an infinite countable set. Its powerset \( 2^S \) is not countable.

**Proof - Diagonalization**

- \( S \) is countable, so it’s elements can be enumerated.
  \( S = \{s_1, s_2, s_3, s_4, s_5, s_6, \ldots \} \)

  An element \( t \in 2^S \) can be represented by a sequence of 0’s and 1’s such that the \( i \)th position in \( t \) is 1 if \( s_i \) is in \( t \), 0 if \( s_i \) is not in \( t \).

  Example, \( \{s_2, s_3, s_5\} \) represented by

  Example, set containing every other element from \( S \), starting with \( s_1 \) is \( \{s_1, s_3, s_5, s_7, \ldots \} \) represented by

  Suppose \( 2^S \) countable. Then we can enumerate all its elements: \( t_1, t_2, \ldots \)

<table>
<thead>
<tr>
<th>( s )</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>( t_4 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>( t_5 )</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>( t_6 )</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>( t_7 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>...</td>
</tr>
</tbody>
</table>
**Theorem** For any nonempty $\Sigma$, there exist languages that are not recursively enumerable.

**Proof:**

- A language is a subset of $\Sigma^*$.
  
  The set of all languages over $\Sigma$ is

**Theorem** There exists a recursively enumerable language $L$ such that $\overline{L}$ is not recursively enumerable.

**Proof:**

- Let $\Sigma = \{a\}$
  
  Enumerate all TM’s over $\Sigma$:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>aa</th>
<th>aaa</th>
<th>aaaa</th>
<th>aaaaa</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(M_1)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>$L(M_2)$</td>
<td>1</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>$L(M_3)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$L(M_4)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>$L(M_5)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>...</td>
</tr>
</tbody>
</table>

...
The next two theorems in conjunction with the previous theorem will show that there are some languages that are recursively enumerable, but not recursive.

**Theorem** If languages $L$ and $\bar{L}$ are both RE, then $L$ is recursive.

**Proof:**

- There exists an $M_1$ such that $M_1$ can enumerate all elements in $L$.
- There exists an $M_2$ such that $M_2$ can enumerate all elements in $\bar{L}$.
- To determine if a string $w$ is in $L$ or not in $L$ perform the following algorithm:

**Theorem:** If $L$ is recursive, then $\bar{L}$ is recursive.

**Proof:**

- $L$ is recursive, then there exists a TM $M$ such that $M$ can determine if $w$ is in $L$ or $w$ is not in $L$. $M$ outputs a 1 if a string $w$ is in $L$, and outputs a 0 if a string $w$ is not in $L$.
- Construct TM $M'$ that does the following. $M'$ first simulates TM $M$. If TM $M$ halts with a 1, then $M'$ erases the 1 and writes a 0. If TM $M$ halts with a 0, then $M'$ erases the 0 and writes a 1.

Hierarchy of Languages:
**Definition** A grammar \( G=(V,T,S,P) \) is *unrestricted* if all productions are of the form

\[
u \rightarrow v\]

where \( u \in (V \cup T)^+ \) and \( v \in (V \cup T)^* \)

**Example:**
Let \( G=(\{S,A,X\},\{a,b\},S,P) \), \( P= \)

\[
\begin{align*}
S & \rightarrow bAaaX \\
bAa & \rightarrow abA \\
AX & \rightarrow \lambda
\end{align*}
\]

**Example** Find an unrestricted grammar \( G \) s.t. \( L(G)=\{a^n b^n c^n | n>0\} \)
\( G=(V,T,S,P) \)
\( V=\{S,A,B,D,E,X\} \)
\( T=\{a,b,c\} \)
\( P= \)

1) \( S \rightarrow AX \)
2) \( A \rightarrow aAbc \)
3) \( A \rightarrow aBbc \)
4) \( Bb \rightarrow bB \)
5) \( Bc \rightarrow D \)
6) \( Dc \rightarrow cD \)
7) \( Db \rightarrow bD \)
8) \( DX \rightarrow EXc \)

There are some rules missing in the grammar.

To derive string \( aabbbccc \), use productions 1,2 and 3 to generate a string that has the correct number of a’s b’s and c’s. The a’s will all be together, but the b’s and c’s will be intertwined.

\[
S \Rightarrow AX \Rightarrow aAbcX \Rightarrow aaAbcbcX \Rightarrow aabBbcbcX
\]
**Theorem** If G is an unrestricted grammar, then L(G) is recursively enumerable.

**Proof:**

- List all strings that can be derived in one step.

List all strings that can be derived in two steps.

**Theorem** If L is recursively enumerable, then there exists an unrestricted grammar G such that L=L(G).

**Proof:**

- L is recursively enumerable.
  \[ \Rightarrow \] there exists a TM M such that L(M)=L.
  \[
  M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)
  \]
  \[ q_0w \xrightarrow{*} x_1q_fx_2 \text{ for some } q_f \in F, \; x_1, x_2 \in \Gamma^* \]
  Construct an unrestricted grammar G s.t. L(G)=L(M).
  \[ S \Rightarrow w \]
  Three steps
  1. \[ S \Rightarrow B \ldots B \# x_q_y B \ldots B \]
     with \( x, y \in \Gamma^* \) for every possible combination
  2. \[ B \ldots B \# x_q_y B \ldots B \Rightarrow B \ldots B \# q_0 w B \ldots B \]
  3. \[ B \ldots B \# q_0 w B \ldots B \Rightarrow w \]
**Definition** A grammar $G$ is *context-sensitive* if all productions are of the form

$$x \rightarrow y$$

where $x, y \in (V \cup T)^+$ and $|x| \leq |y|$.

**Definition** $L$ is context-sensitive (CSL) if there exists a context-sensitive grammar $G$ such that $L=L(G)$ or $L=L(G) \cup \{\lambda\}$.

**Theorem** For every CSL $L$ not including $\lambda$, $\exists$ an LBA $M$ s.t. $L=L(M)$.

**Theorem** If $L$ is accepted by an LBA $M$, then $\exists$ CSG $G$ s.t. $L(M)=L(G)$.

**Theorem** Every context-sensitive language $L$ is recursive.

**Theorem** There exists a recursive language that is not CSL.