Linear programming,
integer linear programming,
mixed integer linear programming

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Example linear program

- maximize $3x + 2y$

subject to

- $4x + 2y \leq 16$
- $x + 2y \leq 8$
- $x + y \leq 5$
- $x \geq 0$
- $y \geq 0$

- We make reproductions of two paintings

- Painting 1 sells for $30, painting 2 sells for $20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red
Solving the linear program graphically

maximize \( 3x + 2y \)

subject to
\( 4x + 2y \leq 16 \)
\( x + 2y \leq 8 \)
\( x + y \leq 5 \)
\( x \geq 0 \)
\( y \geq 0 \)

optimal solution: \( x=3, y=2 \)
Proving optimality

$$\text{maximize } 3x + 2y$$

subject to

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

Recall: optimal solution: $$x=3, y=2$$

Solution value = 9+4 = 13

How do we prove this is optimal (without the picture)?
Proving optimality…

\[
\text{maximize } 3x + 2y \\
\text{subject to } \\
4x + 2y \leq 16 \\
x + 2y \leq 8 \\
x + y \leq 5 \\
x \geq 0 \\
y \geq 0
\]

We can rewrite the blue constraint as
\[
2x + y \leq 8
\]
If we add the red constraint
\[
x + y \leq 5 \\
\text{we get}
\[
3x + 2y \leq 13
\]
Matching upper bound!
(Really, we added .5 times the blue constraint to 1 times the red constraint)
Linear combinations of constraints

\[
\text{maximize } 3x + 2y \\
\text{subject to } \\
4x + 2y \leq 16 \\
x + 2y \leq 8 \\
x + y \leq 5 \\
x \geq 0 \\
y \geq 0
\]

\[
b(4x + 2y \leq 16) + \
g(x + 2y \leq 8) + \
r(x + y \leq 5) = \\
(4b + g + r)x + \
(2b + 2g + r)y \leq \\
16b + 8g + 5r
\]

4b + g + r must be at least 3

2b + 2g + r must be at least 2

Given this, minimize \(16b + 8g + 5r\)
Using LP for getting the best upper bound on an LP

**maximize** $3x + 2y$

**subject to**

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

**minimize** $16b + 8g + 5r$

**subject to**

$4b + g + r \geq 3$

$2b + 2g + r \geq 2$

$b \geq 0$

$g \geq 0$

$r \geq 0$

the dual of the original program

- Duality theorem: any linear program has the same optimal value as its dual!
Modified LP

\[ \text{maximize } 3x + 2y \]
\[ \text{subject to} \]
\[ 4x + 2y \leq 15 \]
\[ x + 2y \leq 8 \]
\[ x + y \leq 5 \]
\[ x \geq 0 \]
\[ y \geq 0 \]

Optimal solution: \( x = 2.5, \ y = 2.5 \)

Solution value = \( 7.5 + 5 = 12.5 \)

Half paintings?
Integer (linear) program

maximize $3x + 2y$

subject to

$4x + 2y \leq 15$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$, integer

$y \geq 0$, integer

optimal LP solution: $x=2.5$, $y=2.5$ (objective 12.5)

optimal IP solution: $x=2$, $y=3$ (objective 12)
Mixed integer (linear) program

\[
\text{maximize } 3x + 2y \\
\text{subject to } \\
4x + 2y \leq 15 \\
x + 2y \leq 8 \\
x + y \leq 5 \\
x \geq 0 \\
y \geq 0, \text{ integer}
\]

optimal LP solution: \(x=2.5, y=2.5\) (objective 12.5)

optimal IP solution: \(x=2, y=3\) (objective 12)

optimal MIP solution: \(x=2.75, y=2\) (objective 12.25)
Exercise in modeling: knapsack-type problem

- We arrive in a room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for $11
  - There are 3 units available
- Unit of object B: 4kg, 4 liters, sells for $4
  - There are 4 units available
- Unit of object C: 6kg, 3 liters, sells for $9
  - Only 1 unit available
- What should we take?
The *general* version of this knapsack IP

maximize $\sum_j p_j x_j$
subject to
$\sum_j w_j x_j \leq W$
$\sum_j v_j x_j \leq V$
(for all $j$) $x_j \leq a_j$
(for all $j$) $x_j \geq 0$, $x_j$ integer
Exercise in modeling: cell phones (set cover)

- We want to have a working phone in every continent (besides Antarctica)
- … but we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E
Exercise in modeling: hot-dog stands

• We have two hot-dog stands to be placed in somewhere along the beach

• We know where the people that like hot dogs are, how far they are willing to walk

• Where do we put our stands to maximize #hot dogs sold? (price is fixed)